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A Novel Generalized Reduced Gradient Optimized Fractional Hausdorff Grey Model for Carbon Dioxide Emission Forecasting

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ABSTRACT

Forecasting CO_2 emissions is a crucial task for environmental management, energy planning and climate change mitigation strategies as it is one of the main drivers of global climate change. The GM(1,1) grey prediction model has been extensively used to predict and solve small sample forecasting problems, but it is usually restricted by the first-order accumulation method and the traditional parameter estimation method, which may not provide the best solution. In view of the above limitations, a Generalized Reduced Gradient Fractional Hausdorff Grey Model (GRG-FHGM(1,1)) is proposed in this paper, which improves on the above shortcomings. In the proposed model, the order of the accumulation is adjusted using the Hausdorff fractional-order accumulation, and the Generalized Reduced Gradient (GRG) nonlinear optimization algorithm is used to optimize the order of the fractional accumulation, the development coefficient and grey input parameter simultaneously. The model's effectiveness was tested with the data of CO_2 emissions from 2010 to 2023 of the International Energy Agency (IEA) for Malaysia. The dataset was divided into training (2010–2019) and testing (2020–2023) periods. GRG-FHGM(1,1)'s forecasting accuracy was tested and matched with four benchmark models: GM(1,1), FHGM(1,1), Linear Regression (LR), and Auto-ARIMA. Experimental results show that the proposed model achieved the lowest testing Mean Absolute Percentage Error (MAPE) of 2.551% and Root Mean Square Error (RMSE) of 7.3705, outperforming GM(1,1) (MAPE = 2.831%, RMSE = 9.0880), FHGM(1,1) (MAPE = 2.831%, RMSE = 9.0880), Linear Regression (MAPE = 2.623%, RMSE = 7.5591), and Auto-ARIMA (MAPE = 6.302%, RMSE = 17.1125). Performance on the training set was the lowest for Auto-ARIMA, but the test set had a much worse MAPE which suggested that this model had poor generalization ability and may be overfitting. The results show that the use of Hausdorff fractional accumulation combined with the parameter optimization with GRG can greatly improve the accuracy of forecasting, robustness and generalization capability. The model presented in this paper, GRG-FHGM(1,1), is a reliable and computational efficient model that can be used for forecasting CO_2 emission for small sample. The results indicate that the model is a suitable decision support tool for environmental planning, carbon management, sustainable development and policy formulation to mitigate climate change.

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1. Introduction

CO_2 Emissions are known as one of the major contributors to climate change and environmental degradation in the world. The International Energy Agency (IEA) estimates that CO_2 emissions amounted to around 33 Gt in the year 2019 [1]. While some developed countries saw a small reduction in emissions because of an increase in the use of renewable energy and a decrease in growth, global carbon emissions will likely continue to grow as a result of future industrialisation, urbanisation and energy use. Due to this, the accurate prediction of CO_2 emission has gained in significance for climate change mitigation policy, carbon reduction policy, sustainable energy policy, and environmental management.

In recent decades, various forecasting methods for CO_2 emission prediction have been proposed, which can be roughly divided into three categories: artificial intelligence (AI)-based methods, classical statistical methods or grey forecasting models, etc. AI methods are especially useful for discovering non-linear trends and relationships in environmental and energy data. There have been several studies using evolutionary neural networks and hybrid optimization-based forecasting models to enhance prediction accuracy and model adaptability [2,3]. Using machine learning approaches such as random forest and particle swarm optimization (PSO) in conjunction with back-propagation (BP) neural networks has also been found to improve forecasting accuracy by tuning the parameters of the model and minimizing the prediction errors [4]. In the meantime, fractional grey prediction models optimized have shown a high application value in dealing with small sample and uncertain datasets, which give better forecasting precision than the grey models [5]. While these improvements have been made, there are still challenges regarding model complexity, interpretation, and forecasting accuracy, and more efficient and adaptive forecasting methods are needed.

In recent times, the main thrust of development in the field of AI forecasting has been machine learning, deep learning, and ensemble forecasting structures. Recently, certain deep learning algorithms like Long Short-Term Memory (LSTM), Gated Recurrent Unit (GRU), Transformer networks, Extreme Learning Machines (ELM), Random Forests (RF), and Ensemble learning models were proven to perform very well in capturing the nonlinear and temporal dependencies present in environmental datasets [6,7]. These models can be used to model complex relationships between economic, environment and energy related variables and can often provide superior prediction of the modeled variables when adequate training data exists. But, in general, AI based models need lots of historical data, hyperparameter tuning and a lot of computing power. They can perform poorly when there is limited data, incomplete data, or faulty data, often seen in environmental forecasting applications [5,6].

Because of their well-established mathematical principles and their interpretability, classical statistical methods are considered important category of forecasting methods for prediction of carbon emissions. They have been used in various fields such as estimating carbon footprint in transportation systems [7] and evaluating emission reduction policies in the energy sector [8]. Unfortunately, statistical models are of limited use because they are very difficult to implement, and they can also lack predictive power due to the assumption of data stationarity, data normality, and linearity between variables. Review studies in recent years have pointed out that these assumptions are often not met in environmental and climate data, which often have complex, nonstationary and nonlinear temporal characteristics [10]. For these reasons, the construction of more flexible forecasting models has been driven by the need to consider the complex nature of carbon emissions processes, and this has led to the growth of advanced models based on learning.For this reason, the development of more flexible forecasting models that can capture the complexity of carbon emission processes has been encouraged, resulting in the development of advanced learning-based forecasting models.

The small sample size, insufficient information and uncertainty are the key features of the problem, which has been attracting the attention of the gray system theory as an alternative forecasting method. Many variants of the GM(1,1) model have been presented since its development [11] to make the model flexible, adaptable and better in predicting the future. In recent years, fractional-order grey model, optimization-based grey model and rolling grey forecasting model have been developed and applied to the fields of carbon emission and greenhouse gas prediction with better forecast accuracy [12-14]. Even though these developments were accomplished, most of the current grey forecasting models are still not satisfactory in describing the complex nonlinear trends and dynamic characteristics of carbon emission data. Furthermore, conventional estimation methods are typically used to choose the model parameters which can be sub-optimal. The limitations underscore the importance of developing more sophisticated grey forecasting models that incorporate effective optimization methods to further improve the forecasting performance and robustness of the grey models.

Even though the classical grey forecasting models are efficient, they suffer from the fact that all the historical observations are given the same importance in the forecasting process, without taking into account the different temporal importance of the historical observations [15]. This assumption might not be valid for dynamic systems where the more recent data may be more relevant and timely than the older data. However, to solve this problem, the fractional-order accumulation (FOA) idea is introduced into grey forecasting model, which makes the weight system more flexible according to the different time attributes of the data [16]. Based on this theoretical improvement, many fractional grey forecasting models have been proposed and applied to a wide range of fields, such as forecasting carbon emission, energy consumption and renewable energy forecasting [17,18]. These models perform better than a traditional grey forecasting model in most cases, but the accuracy of forecasting varies significantly with the choice of fractional accumulation order. Thus, it is still very important to search for an optimum fractional order, which is what has driven further research on optimization-based grey forecasting method.

The development of recent grey forecasting has increasingly focused on the application of the advanced accumulation operator and intelligent optimization algorithm to solve the problems of traditional grey forecasting. Some studies have proposed fractional time-delayed grey models (FTDG), conformable fractional grey models (CFGM), and discrete fractional accumulation grey models (DFAGM) to improve the ability of grey forecasting models to describe the nonlinear and dynamic characteristics of a system [19-21]. Moreover, it has been demonstrated that the improved fractional discrete grey Bernoulli models have better forecasting accuracy in carbon emission prediction applications [22]. While these improvements have greatly improved model accuracy, current methods are mostly concerned with optimizing the order of accumulation of fractions, and still use traditional parameter estimation techniques to calculate the model coefficients. This might compound the model's capacity to optimally solve a global problem and fully take advantage of the forecasting power of fractional accumulation structures. Hence, additional studies are needed to design comprehensive optimization schemes that can optimise parameters of the model and accumulation orders simultaneously for more powerful and precise forecasting results.

Fractional grey models have been proved to have good forecasting ability, but most of the current fractional grey models are based on Gamma-function accumulation operators. Using fractional orders can result in large increase of the computational costs of the Gamma functions, and also cause numerical instability and higher computational costs in the practical forecasting application. Recently, Hausdorff fractional accumulation was introduced into grey forecasting framework as more flexible and efficient approach.To overcome these drawbacks, Hausdorff fractional accumulation is recently introduced as more flexible and efficient approach in grey forecasting frameworks. In many applications including stock price prediction, renewable energy forecasting and modeling of nonlinear systems, various Hausdorff based grey forecasting models have been successfully applied [23-25]. The models have demonstrated their ability to be more adaptive in effective modeling of complex data characteristics with lower computational complexity. More recent have added optimization procedures to Hausdorff fractional grey models, further improving the forecasting accuracy and robustness [26]. The results of these studies indicate that the Hausdorff fractional accumulation operator can be used to achieve a better prediction without the added computational complexity of using a conventional Gamma-function based fractional accumulation operator.

Forecasting research has been growing in the direction of hybrid forecasting frameworks that make use of several modeling constructs to leverage the strengths of each of them. Regarding the hybrid modelling approach, models combining grey system theory, artificial intelligence, machine learning algorithms, and optimization methods have received much attention because they can solve the problems of uncertainty, nonlinearity, and time variability in complex systems. In recent years, the hybrid grey-AI model has been proved to have better forecasting ability than the traditional grey model and traditional statistical methods, effectively improving the adaptability and learning ability of the model [27]. Additionally, survey reviews have pointed out the importance of hybrid intelligent forecasting systems as one of the emerging trends, owing to its ability to enhance the accuracy, robustness, explainability and generalization of the forecasting systems for a variety of applications [10,24,25]. While these benefits exist, the complexity of hybrid frameworks can also bring about parameters tuning, model interpretability, and computational efficiency issues. Thus, high accuracy forecasting models are needed that are both easy to implement and cost-effective, yet not overly complex.

Although great developments have been made in the field of man-made intelligence (AI) and grey forecasting and blended forecasting models, there are still some significant problems that need to be addressed. First, most AI-based and hybrid forecasting methods need to access massive amounts of training data and computational power, which can be especially challenging in forecasting environments with limited data availability. Second, the existing fractional grey forecasting models are mostly still using the accumulation operators derived by the Gamma function, which may result in the higher computation complexity and numerical instability of the model in the procedure of model implementation. Third, most grey forecasting models adopt the method of ordinary least squares (OLS) to calculate model parameters, and background values [28,29]. Background values, however, are subjective determinations and can introduce estimation bias which will influence forecasts. Moreover, parameter estimation based on OLS is not guaranteed to yield the optimal solutions everywhere, especially with nonlinear forecasting problems involving complex data structures.

To overcome these shortcomings, this paper introduces a Generalized Reduced Gradient Fractional Hausdorff Grey Model (GRG-FHGM(1,1)). The proposed model adopts Hausdorff fractional accumulation as the weighting mechanism for historical observations, which has the advantage of avoiding the complexity of the commonly used Gamma function-based accumulation, but is still a flexible and efficient weighting method. Moreover, the Generalized Reduced Gradient (GRG) nonlinear optimization algorithm is used to optimize the model parameters and the fractional accumulation order simultaneously, without using any background value estimation. The proposed model is expected to improve the accuracy of the forecasting, reduce the computational time, and make the model more robust by combining the Hausdorff fractional accumulation with the nonlinear optimization. These annual CO_2 emission data are then used to assess the effectiveness of the proposed GRG-FHGM(1,1) model, and to compare its prediction performance with some existing grey forecasting models, in order to demonstrate that the proposed model has excellent prediction performance and can be used practically for long-term CO_2 emission forecasting.

2. Construction of GRG-FHGM (1,1)

This section starts with a review of conventional GM(1,1) model and Fractional Hausdorff Grey Model (FHGM(1,1)). Then, the mathematical description, the estimation of the parameters and the solution algorithm of the proposed GRG-FHGM(1,1) model are explained in detail.

2.1 GM(1,1) model

Assume $x^{(0)} = \left\{x_1^{(0)}, x_2^{(0)}, \dots, x_k^{(0)}, \dots, x_n^{(0)}\right\}$ represents the original time series data, and $x^{(1)}$is the one-time accumulated generating operation (1-AGO),

$$x_k^{(1)} = \sum_{i=1}^{k} x_i^{(0)},\ k = 1, 2, \dots, n \tag{1}$$

The grey first-order differential equation of the GM(1,1) model is given by

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \tag{2}$$

This can be expressed in a discrete differential form as

$$x_k^{(0)} + az_k^{(1)} = b, \forall k = 2, 3, \dots, n. \tag{3}$$

where $z_k^{(1)} = 0.5x_k^{(1)} + 0.5x_{k-1}^{(1)}, \forall k = 2, 3, \dots, n.$ The values of *a* and *b* are estimated using the least-squares method as

$$\widehat{W} = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T Y \tag{4}$$

where $W = [a, b]^T$, $Y = \left[\ x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}\right]^T$ and $B = \begin{bmatrix} -z_2^{(1)} & -z_3^{(1)} & \cdots & -z_n^{(1)} \\ 1 & 1 & \cdots & 1 \end{bmatrix}^T$

The solution to Eq. (3) can be obtained as:

$$\hat{x}_k^{(1)} = \left(x_1^{(0)} - \frac{b}{a}\right) e^{-a(k-1)} + \frac{b}{a} \tag{5}$$

The predicted values of the GM(1,1) model are:

$$\hat{x}^{(0)} = \begin{cases} \hat{x}_1^{(0)} = \hat{x}_1^{(1)} \\ \hat{x}_k^{(0)} = \hat{x}_k^{(1)} - \hat{x}_{k-1}^{(1)} \end{cases},\ \ k = 1,2, \dots, n \tag{6}$$

2.1 Fractional Hausdorff Grey Model (FHGM)l

The Hausdorff fractional-order accumulation operator assigns different weights to historical observations through the term $[i^r-(i-1)^r]$. Unlike the conventional first-order accumulation operator, this mechanism provides greater flexibility in emphasizing the contribution of individual observations, thereby improving the model's ability to capture dynamic patterns in the data.

Assume $x^{(0)}=\{x^{(0)}(1),x^{(0)}(2),\dots,x^{(0)}(n)\}$ is a nonnegative sequence; its *r*-order accumulative sequence is then given by $x^{(r)}=\{x^{(r)}(1),x^{(r)}(2),\dots,x^{(r)}(n)\}$, the Hausdorff fractional-order accumulation operator [26], given by

$$x^r(k)=\sum_{i=1}^{k}x^{(0)}(i)[i^r-(i-1)^r] \quad (7)$$

The Fractional Hausdorff Grey Model (FHGM (1,1)) model is given by

$$\frac{dx^{(r)}}{dt}+ax^{(r)}=b \quad (8)$$

This can be expressed in a discrete differential form as

$$x_k^{(r)}-x_{k-1}^{(r)}+az_k^{(r)}=b, \forall k=2,3,\dots,n. \quad (9)$$

where $z_k^{(r)}=0.5x_k^{(r)}+0.5x_{k-1}^{(r)}, \forall k=2,3,\dots,n.$ The values of *a* and *b* are estimated using the least-squares method as

$$\widehat{W}=\begin{bmatrix}a\\ b\end{bmatrix}=(B^TB)^{-1}B^TY \quad (10)$$

where $W=[a,b]^T$, $Y=\begin{pmatrix}x^{(r)}(2)-x^{(r)}(1)\\ x^{(r)}(3)-x^{(r)}(2)\\ \vdots\\ x^{(r)}(n)-x^{(r)}(n-1)\end{pmatrix}$ and $B=\begin{pmatrix}-z^{(r)}(2) & 1\\ -z^{(r)}(3) & 1\\ \vdots & \vdots\\ -z^{(r)}(n) & 1\end{pmatrix}$

The solution to Eq. (8) can be obtained as:

$$\hat{x}_k^{(r)}=\left(x_1^{(0)}-\frac{b}{a}\right)e^{-a(k-1)}+\frac{b}{a} \quad (11)$$

The predicted value of FHNGBM(1,1) is

$$\hat{x}^{(0)}(k+1)=(\hat{x}^{(r)}(k+1)-\hat{x}^{(r)}(k))/[k^r-(k-1)^r] \ , \ k=1,2,\dots,n-1 \quad (12)$$

where $\hat{x}^{(1)}(1)=\hat{x}^{(0)}(1)$.

2.3 GRG-Fractional Hausdorff Grey Model (FHGM)

Based on the Hausdorff fractional accumulation operator, the Fractional Hausdorff Grey Model, denoted as FHGM(1,1), can be expressed by the following first-order differential equation:

$$\frac{dx^{(r)}(t)}{dt}+a(t)=b \tag{13}$$

Therefore, the solution to Eq. (13) is

$$\hat{x}_k^{(r)}=\left(x_1^{(0)}-\frac{b}{a}\right)e^{-a(k-1)}+\frac{b}{a} \tag{14}$$

The predicted value of GRG-FHNGBM(1,1) is

$$\hat{x}^{(0)}(k+1)=(\hat{x}^{(r)}(k+1)-\hat{x}^{(r)}(k))/[k^r-(k-1)^r] \tag{15}$$

where $\hat{x}^{(1)}(1)=\hat{x}^{(0)}(1)$.

There are several unknown parameters in the proposed GRG-FHGM(1,1) model: a is the development coefficient, b is the grey input and r is the fractional accumulation order. The parameters are not measurable and an optimization strategy is used to find the near-optimal values.
In this study, the Generalized Reduced Gradient (GRG) nonlinear optimization algorithm is used in Microsoft Excel Solver to estimate the values of the model parameters. The GRG method is used since it can be effectively applied to a nonlinear constrained optimization problem and is able to find good solutions efficiently. The fractional order (r) lies in the range (0 < r < 1) and the initial values of the parameters a and b are calculated from Eq. (4).

The optimization goal is to minimise the Mean Absolute Percentage Error (MAPE) between the observed and the predicted values. Thus, an optimization problem can be posed as follows:

$$\min_{r,a,b} MAPE=\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right|\times 100\% \tag{16}$$

where $x^{(0)}(k)$ and $\hat{x}^{(0)}(k)$ denote the actual and predicted values at time, *k*, respectively, and *n* represents the total number of observations.
The following summarizes the parameter estimation procedures of the benchmark models for comparison's sake:

i. The development coefficient, *a*, and the grey input, *b* are obtained by Ordinary Least Squares (OLS) method.
ii. FHGM(1,1): the parameters *a* and *b* are estimated by the OLS method, the fractional accumulation order (r) is optimized in the range (0 < *r* < 1) using the GRG nonlinear algorithm to minimize MAPE.
iii. By optimizing the parameters *a*, *b* and *r* using GRG nonlinear algorithm to minimize the MAPE, the intervention of the conventional background-value estimation is avoided and the forecasting accuracy is improved.

3. Applications of CO_2 Emissions Forecasting

Addressing the negative effects of CO_2 emissions on environmental sustainability and economic development is still a major challenge worldwide. This is because it is imperative to have accurate forecasting of future CO_2 emissions to help support carbon neutrality efforts, climate mitigation measures, and long-term energy planning. Accurate predictions can give policy makers useful information that they can use to develop effective environmental laws and emission reduction policies.

The effectiveness of the proposed model was tested by using the annual CO_2 emission data for Malaysia between 2010 to 2023 which were taken from the International Energy Agency (IEA) [1]. Malaysia was chosen as the country for the case study because of its fast-growing economy and the increasing amount of energy used in the country, which are responsible for a large proportion of carbon emissions generated by the country. The dataset consists of 14 annual observations and the data from 2010 to 2019 were used as the training and parameter estimation data, while the data from 2020 to 2023 were used as out-of-sample data for model testing and validation.

The proposed GRG-FHGM(1,1) model has three parameters, development coefficient *a*, grey input parameter *b* and fractional accumulation order *r*. Conventional grey forecasting model parameters are estimated by one by one whereas these parameters are estimated together with the Generalized Reduced Gradient (GRG) nonlinear optimization algorithm in MS-Excel Solver. The optimization process is designed to obtain the best possible values for *a*, *b* and *r*, to minimize the Mean Absolute Percentage Error (MAPE). The fractional order ranged between ($0 < r < 1$) and the initial values of a and b were determined with the Ordinary Least Squares (OLS) method.

The parameter estimation procedures used differ for each of the models used for benchmarking. The parameters *a* and *b* are directly estimated in the GM(1,1) model by the OLS method. For FHGM(1,1), *a* and *b* are estimated with OLS first and then *r* is optimized with GRG. The proposed GRG-FHGM(1,1) optimizes all parameters of the model in one framework, which allows for searching for the optimal solution in a more comprehensive manner.

The optimum values of the parameters and training MAPE for GM(1,1), FHGM(1,1) and GRG-FHGM(1,1) models are given in Table 1. The model proposed in this study was able to obtain the lowest MAPE of 2.655%, greater than the results of GM(1,1) and FHGM(1,1), with MAPE values of 2.793% and 2.793%, respectively. The differences between the estimated parameter values seem quite small but the optimal combination of the parameters gave a significant improvement in forecasting accuracy.

The optimised fractional order, *r*, for both FHGM(1,1) and GRG-FHGM(1,1) approaches have been found to be around 1.0, showing that the accumulated sequence preserves much information from the original sequence, and has a flexibility that is provided by fractional accumulation. Although the fractional orders are similar, the GRG optimization framework was able to obtain improved estimates of *a* and *b*, and thus reduce the forecasting errors when compared to traditional estimation procedures. The result demonstrated the potential of the GRG algorithm to detect a better set of parameters for modelling trends on CO_2 emission for Malaysia.

In general, the results show that the proposed GRG-FHGM(1,1) model is effective in improving the performance of forecasting in the model by combining Hausdorff fractional accumulation and nonlinear optimization. The model keeps the good properties of grey system theory for small sample forecasting and enhances the accuracy of the parameter estimation and the forecasting ability. The optimized parameters derived from the training stage values were then used for generating out-of-sample forecasts for the testing stage and the robustness and practical applicability of the proposed model were subsequently assessed using MAPE and RMSE performance measures.

Table 1
Place The model parameters and MAPE of the training data of the GM(1,1), FHGM(1,1) and GRG-FHGM(1,1) models

Parameter	GM(1,1)	FHGM(1,1)	GRG-FHGM(1,1)
a	220.19393	220.19391	219.99969
b	-0.01820	-0.01820	-0.01713
r	-	0.99999	0.99999
MAPE%	2.79294	2.79303	2.65529

4. Results and Discussion

The effectiveness of the proposed GRG-FHGM(1,1) model was tested by comparing its forecasting performance with four benchmark models: GM(1,1), FHGM(1,1), Auto-ARIMA and Linear Regression (LR). The measures of forecasting accuracy were indicated by Mean Absolute Percentage Error (MAPE) and Root Mean Square Error (RMSE), which are listed in Table 2. The results clearly demonstrate that the overall performance of GRG-FHGM(1,1) was the best with the lowest testing MAPE (2.551%) and RMSE (7.3705) among all models. The testing error was also significantly higher with Auto-ARIMA (4.199%), reflecting the fact that good results on the historical data does not guarantee good results on the data to be forecasted. The proposed model, on the other hand, yielded consistently low error values for both the training set and the testing set, indicating that it provides stable and reliable predictions.

Table 2
Comparative analysis of GRG-FHGM(1,1), GM(1,1), FHGM(1,1), Linear Regression and Auto-ARIMA in terms of MAPE and RMSE

	MAPE (%)		**RMSE**	
Model	**Training**	**Testing**	**Training**	**Testing**
GM(1,1)	2.793	2.831	8.275	9.088
FHGM(1,1)	2.793	2.831	8.275	9.088
GRG-FHGM(1,1)	2.655	**2.551**	**8.034**	**7.371**
Auto-ARIMA	**2.170**	6.302	9.543	17.113
Linear Regression	2.764	2.623	8.034	7.559

One thing that is critical here is that all the models were derived from the same data, and there are only fourteen annual observations. Despite the limited amount of data available, GRG-FHGM(1,1) was able to outperform all benchmark methods. This discovery is one of the major advantages of gray forecasting model: it can give accurate forecast when information is incomplete or limited. This is useful for long-term and extensive environmental forecasting and is especially valuable when long-term data are not available. The proposed model can generate high forecasting accuracy with a limited number of observations unlike many machine learning and deep learning methods which need a large number of training data.

This improvement in the performance of GRG-FHGM(1,1) is due to the synergistic effect of the Hausdorff fractional accumulation and Generalized Reduced Gradient (GRG) optimization. The Hausdorff fractional accumulation mechanism enables the model to make better use of historical data, by giving different weights to data from the past, according to a flexible importance function. The GRG algorithm optimizes the model parameters simultaneously and at the same time, the forecasting structure will better reflect the characteristics of the data. These factors jointly contribute to better model performance at simulating the evolution of the CO_2 emission time series and to minimizing forecasting errors.

The results also show that parameter optimization plays an important role in grey forecasting models. The models, GM(1,1), FHGM(1,1) and GRG-FHGM(1,1), are similar in modelling principles but their forecasting performances are noticeably different. Interestingly, GM(1,1) and FHGM(1,1) gave close testing results, with the latter method showing a slight improvement in forecasting, implying that improvements in forecasting by introducing Hausdorff fractional accumulation alone are not significant. However, the benefits of the fractional accumulation mechanism can only be realised if supported by good parameter optimisation. This is one of the reasons that the proposed GRG-FHGM (1,1) model gave the best forecasting accuracy in comparison with the other grey models.

The forecasting trajectories depicted in Fig. 1 are also a good indication of the effectiveness of the proposed model. Forecasts produced by GRG-FHGM(1,1) closely mimic the actual CO_2 emission trend during the training and testing period. This means that the model can explain the general long-term structure of the emission series and be sensitive to more recent changes in the emission series. The balance is particularly important in the context of carbon emission forecasting because short-term variations in emissions are driven by economic conditions, industry, technology and environmental policies as well as long-term development trends.

In contrast, Auto-ARIMA demonstrated good fitting performance during the training period; however, it had poor forecasting ability during the testing period. This is an illustrated example of a well known weakness of ARIMA-type models, which are heavily dependent on historical statistical relationships and linearity assumptions. In reality, carbon emission systems are influenced by a variety of economic, technological and policy-driven elements that can create nonlinear patterns and structural shifts. Therefore, models that rely on the autoregressive and moving-average processes alone can lack adaptability to changing conditions in the future relative to past conditions.

In a general sense, the results show that the grey system theory is still applicable to predicting uncertainty problems with small samples and partial information. The proposed GRG-FHGM(1,1) model is a new nonlinear optimization-based grey forecasting model, which combines a flexible accumulation mechanism with a powerful nonlinear optimization approach. The integration allows the model to leverage the information contained in the limited data to better estimate parameters, and to make more useful use of that information.

A remarkable result is the good stability of the proposed model. The training error is not significantly larger than the testing error, which indicates that GRG-FHGM(1,1) has a good ability to generalize to unseen data and is not likely to be overfitting. This feature is especially valuable for applications in the field of environmental forecasting which may involve unforeseen changes in policy, technology or economic behavior. Under these conditions a model that has low training error may not be as useful as a model that does well on new data.

In conclusion, the results show that GRG-FHGM(1,1) offers a strong, reliable, and efficient tool for CO_2 emission forecasting. The model has the ability to effectively integrate Hausdorff fractional accumulation and GRG optimization, thus improving the accuracy of forecasting, stability, and the predictive ability of the model. Based on these advantages, it is believed that the proposed model is a potential alternative for forecast carbon emission and can be successfully used to other complex and uncertain small-sample forecasting problems as well.

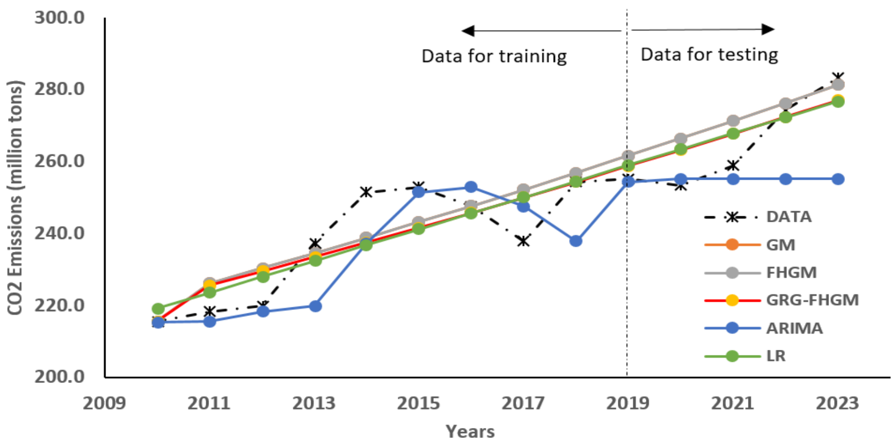

Fig. 1. Comparison of actual CO_2 emissions and forecasts generated by GM(1,1), FHGM(1,1), GRG-FHGM(1,1), Auto-ARIMA, and Linear Regression Models

5. Conclusions

As the accurate forecasting of CO_2 emission is critical for environment sustainability, energy planning and mitigating climate change. This study put forward a new forecasting model, the Generalized Reduced Gradient Fractional Hausdorff Grey Model (GRG-FHGM(1,1)), based on the Hausdorff fractional-order accumulation and Generalized Reduced Gradient (GRG) nonlinear optimization in grey system modelling. The proposed approach simultaneously optimizes the development coefficient, grey input parameter and fractional accumulation order, which is different from the conventional GM(1,1) and FHGM(1,1) models, and then the performance of the parameter estimation and forecasting can be improved.

Malaysia's annual CO_2 emission data for the period of 2010 to 2023 was used for the evaluation of the proposed model. GM(1,1), FHGM(1,1), Linear Regression (LR), and Auto-ARIMA models were compared with each other in terms of forecasting performance, and the performance was measured using Mean Absolute Percentage Error (MAPE) and Root Mean Square Error (RMSE). The results showed that the overall performance of GRG-FHGM(1,1) was the best with the lowest testing MAPE of 2.551% and RMSE of 7.3705. Although the lowest training error was obtained by Auto-ARIMA, its forecasting ability was found to be poor during testing, meaning that it had poor generalization ability. The proposed model, however, showed relatively low forecasting errors in both training and testing sets, demonstrating the robustness and reliability of the model for future forecasting.

The results also showed that the optimization of parameters is important to enhance the accuracy of grey forecasting. Compared with the traditional forecasting models GM(1,1) and FHGM(1,1), the simultaneous optimization of the model parameters obtained more accurate forecasting results. This outcome suggests that forecasting accuracy is highly dependent on the choice of the parameters, and that good optimisation can dramatically enhance forecasting accuracy.

The graphical analysis also showed that the proposed model was able to well capture the long-term trend and short term fluctuations of CO_2 emissions in Malaysia. The predicted trajectory was also closely aligned with the emission history during the training and testing phases, showing good performance in adjusting to the most recent observations. It is a capability that is of special significance for environmental forecasting, as emission patterns can be driven by long-term developmental trends and also short-term economic, technological, and policy-related changes.

In practice, the suggested GRG-FHGM(1,1) model is a useful decision-support tool for governments, environmental agencies, and policy makers. Precise estimates of emissions can help assess reductions in carbon emissions, energy transition strategies and track progress towards sustainability targets. Overall, the study shows that GRG-FHGM(1,1) is a valid compromise between the model stability and the forecasting accuracy, while being computationally efficient. Future studies can expand the model to include other explanatory variables and/or incorporate the model into machine learning and hybrid intelligent forecasting methods to improve forecasts even more.

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